Computers and Mathematics with Applications 61 (2011) 851–859

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On the convergence of hybrid projection algorithms for asymptotically quasi-ϕ-nonexpansive mappings

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ARTICLE INFO

Article history:
Received 9 September 2009
Received in revised form 14 December 2010
Accepted 14 December 2010

Keywords:
Asymptotically quasi-ϕ-nonexpansive mapping
Fixed point
Generalized projection
Quasi-ϕ-nonexpansive mapping

ABSTRACT

Using the Mann iteration in an infinite-dimensional Hilbert space, only weak convergence theorems are obtained even for nonexpansive mappings. The purpose of this paper is to modify the Mann iteration and prove the strong convergence theorems without any compactness assumption for asymptotically quasi-ϕ-nonexpansive mappings. Moreover, strong convergence theorems are also established in a uniformly smooth and strictly convex Banach space with the Kadec–Klee property.

1. Introduction and preliminaries

Let E be a real Banach space, C a nonempty subset of E and $T : C \to C$ a nonlinear mapping. The mapping T is said to be *asymptotically regular* on C if for any bounded subset K of C,

$$\limsup_{n\to\infty}\{\|T^{n+1}x - T^n x\| : x \in K\} = 0.$$

The mapping T is said to be *closed* if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. A point $x \in C$ is said to be a *fixed point* of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T and use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Recall that T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

T is said to be *quasi-nonexpansive* if $F(T) \ne \emptyset$ and

$$\|x - Ty\| \le \|x - y\|, \quad \forall x \in F(T),\ \forall y \in C.$$

T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n\|x - y\|, \quad \forall x, y \in C,\ \forall n \ge 1.$$

T is said to be *asymptotically quasi-nonexpansive* if $F(T) \ne \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|x - T^n y\| \le k_n\|x - y\|, \quad \forall x \in F(T),\ \forall y \in C,\ \forall n \ge 1.$$

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2010.12.033

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. In uniformly convex Banach spaces, they proved that if C is nonempty bounded closed and convex, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the fixed point set of T is closed and convex. Since 1972, a host of authors have studied the weak and strong convergence theorems of iterative algorithms for such a class of mappings.

Recall that the Mann iteration was introduced by Mann [2] in 1953. Since then, the constructions of fixed points for nonexpansive mappings via the Mann iteration has been extensively investigated by many authors. The Mann iteration generates a sequence $\{x_n\}$ in the following manner.

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily,} \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0, 1)$.

If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iteration (1.1) converges weakly to a fixed point of T; see [3,4].

In 1991, Schu [5] gave an adaptation of the Mann iteration for asymptotically nonexpansive mappings as follows:

$$\begin{cases} x_1 \in C \quad \text{chosen arbitrarily,} \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in the interval $(0, 1)$. Weak convergence theorems are established under certain restrictions imposed on the control sequence $\{\alpha_n\}$ in the framework of real Hilbert spaces; see also [6].

It is well known that, in an infinite-dimensional Hilbert space, only weak convergence theorems for the Mann iteration were established even for nonexpansive mappings. Attempts to modify the Mann iteration for nonexpansive mappings and asymptotically nonexpansive mappings by hybrid projection algorithms have recently been made so that strong convergence theorems are obtained; see, for example, [7–13] and the references therein. Nakajo and Takahashi [10] proposed the following modification of the Mann iteration for one single nonexpansive mapping T in a Hilbert space. To be more precisely, they proved the following theorem:

Theorem NT. *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such that $\alpha_n \leq 1-\delta$ for some $\delta \in (0, 1]$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the following algorithm.*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.3}$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Subsequently, a number of authors improved Nakajo and Takahashi's results in different directions. In 2006, Kim and Xu [8] extended Theorem NT to the class of asymptotically nonexpansive mappings and strong convergence theorems are established in the framework of Hilbert spaces as follows.

Theorem KX. *Let C be a bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\alpha_n \leq a$ for all n and for some $0 < a < 1$. Define a sequence $\{x_n\}$ in C by the following algorithm.*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\| + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.4}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(\mathrm{diam} C)^2 \to 0 \quad \text{as } n \to \infty.$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

In 2008, Takahashi et al. [13], still in a Hilbert space, improved Theorem NT by using the so-called shrinking projection methods for nonexpansive mappings as the following.

Theorem TTK. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows.*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 0, \end{cases} \tag{1.5}$$

where $0 \le \alpha_n \le a < 1$ for all $n \ge 0$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Recently, many authors further considered the problem of modifying the Mann iteration in the framework of real Banach spaces; see [14–28] and the references therein. Before proceeding further, we will recall some definitions and propositions in Banach spaces.

Let E be a Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \left\|\frac{x_n+y_n}{2}\right\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U_E$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E has the *Kadec–Klee property* if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on the Kadec–Klee property, the readers can refer to [29–31] and the references therein. It is well known that if E is a uniformly convex Banach space, then E satisfies the Kadec–Klee property.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [32] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{1.6}$$

Observe that, in a Hilbert space H, (1.6) is reduced to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$. The *generalized projection* $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \inf_{y\in C} \phi(y, x)$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, for example, [33,32,29,31]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{1.7}$$

Remark 1.1. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x, y) = 0$ then $x = y$. From (1.7), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, one has $Jx = Jy$. Therefore, we have $x = y$; see [29,31] for more details.

Let C be a nonempty closed convex subset of E and T a mapping from C into itself. A point p in C is said to be an *asymptotic fixed point* of T [34] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is said to be *relatively nonexpansive* if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The mapping T is said to be *relatively asymptotically nonexpansive* if $\widetilde{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, Tx) \le k_n\phi(p, x)$ for all $x \in C$, $p \in F(T)$ and $n \ge 1$. The asymptotic behavior of a relatively nonexpansive mappings was studied in [35–37].

The mapping T is said to be *ϕ-nonexpansive* if $\phi(Tx, Ty) \le \phi(x, y)$ for all $x, y \in C$. T is said to be *quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The mapping T is said to be *asymptotically ϕ-nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(Tx, Ty) \le k_n\phi(x, y)$ for all $x, y \in C$. T is said to be *asymptotically quasi-ϕ-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p, Tx) \le k_n\phi(p, x)$ for all $x \in C$, $p \in F(T)$ and $n \ge 1$.

Remark 1.2. The class of (asymptotically) quasi-ϕ-nonexpansive mappings is more general than the class of relatively (asymptotically) nonexpansive mappings which requires the restriction: $F(T) = \widetilde{F}(T)$.

Remark 1.3. In the framework of Hilbert spaces, (asymptotically) quasi-ϕ-nonexpansive mappings are reduced to (asymptotically) quasi-nonexpansive mappings.

Recently, Matsushita and Takahashi [18] improved Theorem NT from Hilbert spaces to Banach spaces as follows:

Theorem MT. *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{1.8}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $P_{F(T)}x$, where $P_{F(T)}$ is the generalized projection from C onto $F(T)$.

In this paper, motivated by Theorems KX, MT, NT and TTK, we modify the Mann iteration for asymptotically quasi-ϕ-nonexpansive mappings to obtain strong convergence theorems in real Banach spaces without any compactness assumption by using shrinking projection methods. The results presented in this paper improve the corresponding results in [8,18,10,13].

In order to prove our main results, we need the following lemmas.

Lemma 1.4 (*Alber [32]*)**.** *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \ge 0 \quad \forall y \in C.$$

Lemma 1.5 (*Alber [32]*)**.** *Let E be a reflexive, strictly convex and smooth Banach space, C a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x) \quad \forall y \in C.$$

2. Main results

Theorem 2.1. *Let E be a uniformly smooth and strictly convex Banach space which enjoys the Kadec–Klee property and C a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and asymptotically quasi-ϕ-nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Assume that T is asymptotically regular and $F(T)$ is bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)JT^n x_n], \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{cases} \tag{2.1}$$

where $M_n = \sup\{\phi(z, x_n) : z \in F(T)\}$ for each $n \ge 1$. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}$ is the generalized projection from E onto $F(T)$.

Proof. First, we show that $F(T)$ is closed and convex so that $\Pi_{F(T)}x_0$ is well defined. It is easy to check that the closedness of $F(T)$ can be deduced from the closedness of T. We mainly show that $F(T)$ is convex. To this end, for arbitrary $p_1, p_2 \in F(T)$, $t \in (0, 1)$. Putting $p_3 = tp_1 + (1-t)p_2$, we prove that $Tp_3 = p_3$. Indeed, from the definition of ϕ, we see that

$$\begin{aligned} \phi(p_3, T^n p_3) &= \|p_3\|^2 - 2\langle p_3, J(T^n p_3)\rangle + \|T^n p_3\|^2 \\ &= \|p_3\|^2 - 2\langle tp_1 + (1-t)p_2, J(T^n p_3)\rangle + \|T^n p_3\|^2 \\ &= \|p_3\|^2 - 2t\langle p_1, J(T^n p_3)\rangle - 2(1-t)\langle p_2, J(T^n p_3)\rangle + \|T^n p_3\|^2 \\ &= \|p_3\|^2 + t\phi(p_1, T^n p_3) + (1-t)\phi(p_2, T^n p_3) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\ &\le \|p_3\|^2 + k_n t\phi(p_1, p_3) + k_n(1-t)\phi(p_2, p_3) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\ &= (k_n - 1)\big(t\|p_1\|^2 + (1-t)\|p_2\|^2 - \|p_3\|^2\big). \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \phi(p_3, T^n p_3) = 0. \tag{2.2}$$

From (1.7), we see that

$$\lim_{n\to\infty} \|T^n p_3\| = \|p_3\|. \tag{2.3}$$

It follows that

$$\lim_{n\to\infty} \|J(T^n p_3)\| = \|Jp_3\|. \tag{2.4}$$

Since E^* is reflexive, we may, without loss of generality, assume that $J(T^n p_3) \rightharpoonup e^* \in E^*$. In view of the reflexivity of E, we have $J(E) = E^*$. This shows that there exists an element $e \in E$ such that $Je = e^*$. It follows that

$$\begin{aligned}\phi(p_3, T^n p_3) &= \|p_3\|^2 - 2\langle p_3, J(T^n p_3)\rangle + \|T^n p_3\|^2 \\ &= \|p_3\|^2 - 2\langle p_3, J(T^n p_3)\rangle + \|JT^n p_3\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the above equality, we obtain that

$$\begin{aligned}0 &\geq \|p_3\|^2 - 2\langle p_3, e^*\rangle + \|e^*\|^2 \\ &= \|p_3\|^2 - 2\langle p_3, Je\rangle + \|Je\|^2 \\ &= \|p_3\|^2 - 2\langle p_3, Je\rangle + \|e\|^2 \\ &= \phi(p_3, e).\end{aligned}$$

This implies that $p_3 = e$, that is, $Jp_3 = e^*$. It follows that $J(T^n p_3) \rightharpoonup Jp_3 \in E^*$. In view of the Kadec–Klee property of E^* and (2.4), we arrive at

$$\lim_{n\to\infty} \|J(T^n p_3) - Jp_3\| = 0.$$

Since $J^{-1} : E^* \to E$ is demi-continuous, we see that $T^n p_3 \rightharpoonup p_3$. By virtue of the Kadec–Klee property of E and (2.3), we arrive at $T^n p_3 \to p_3$ as $n \to \infty$. Hence

$$TT^n p_3 = T^{n+1} p_3 \to p_3$$

as $n \to \infty$. In view of the closedness of T, we can obtain that $p_3 \in F(T)$. This shows that $F(T)$ is convex.

Next, we show that C_n is closed and convex for each $n \geq 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_h is closed and convex for some h. For $z \in C_h$, we see that $\phi(z, y_h) \leq \phi(z, x_h) + (k_h - 1)M_h$ is equivalent to

$$2\langle z, Jx_h - Jy_h\rangle \leq \|x_h\|^2 - \|y_h\|^2 + (k_h - 1)M_h.$$

Hence C_{h+1} is closed and convex. Then, for each $n \geq 1$, C_n is closed and convex. Now, we are in a position to show that $F(T) \subset C_n$ for each $n \geq 1$. First, we have $F(T) \subset C_1 = C$. Suppose that $F(T) \subset C_h$ for some h. Then, for $\forall w \in F(T) \subset C_h$, we have

$$\begin{aligned}\phi(w, y_h) &= \phi(w, J^{-1}[\alpha_h Jx_h + (1-\alpha_h)JT^h x_h]) \\ &= \|w\|^2 - 2\langle w, \alpha_h Jx_h + (1-\alpha_h)JT^h x_h\rangle + \|\alpha_h Jx_h + (1-\alpha_h)JT^h x_h\|^2 \\ &\leq \|w\|^2 - 2\alpha_h\langle w, Jx_h\rangle - 2(1-\alpha_h)\langle w, JT^h x_h\rangle + \alpha_h\|x_h\|^2 + (1-\alpha_h)\|T^h x_h\|^2 \\ &= \alpha_h\phi(w, x_h) + (1-\alpha_h)\phi(w, T^h x_h) \\ &\leq \alpha_h\phi(w, x_h) + (1-\alpha_h)k_h\phi(w, x_h) \\ &= \phi(w, x_h) + (1-\alpha_h)(k_h - 1)\phi(w, x_h) \\ &\leq \phi(w, x_h) + (k_h - 1)M_h,\end{aligned}$$

which shows that $w \in C_{h+1}$. This implies that $F(T) \subset C_n$ for each $n \geq 1$. On the other hand, it follows from Lemma 1.5 that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(w, x_0) - \phi(w, x_n) \leq \phi(w, x_0),$$

for each $w \in \mathcal{F} \subset C_n$ and for each $n \geq 1$. Therefore the sequence $\phi(x_n, x_0)$ is bounded. From (1.7), we see that the sequence $\{x_n\}$ is also bounded. Since the space E is reflexive, we may, without loss of generality, assume that $x_n \rightharpoonup \bar{x}$. Note that C_n is closed and convex for each $n \geq 1$. It is easy to see that $\bar{x} \in C_n$ for each $n \geq 1$. On the other hand, we have

$$\phi(x_n, x_0) \leq \phi(\bar{x}, x_0).$$

It follows that

$$\phi(\bar{x}, x_0) \leq \liminf_{n\to\infty} \phi(x_n, x_0) \leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \phi(\bar{x}, x_0).$$

This implies that

$$\lim_{n\to\infty} \phi(x_n, x_0) = \phi(\bar{x}, x_0).$$

Hence, we have $\|x_n\| \to \|\bar{x}\|$ as $n \to \infty$. In view of the Kadec–Klee property of E, we obtain that $x_n \to \bar{x}$ as $n \to \infty$.

Next, we show that $\bar{x} \in F(T)$. By the construction of C_n, we have $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$. It follows that

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\ &\le \phi(x_{n+1}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0). \end{aligned} \tag{2.5}$$

Letting $n \to \infty$ in (2.5), we obtain that $\phi(x_{n+1}, x_n) \to 0$. In view of $x_{n+1} \in C_{n+1}$, we arrive at

$$\phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n) + (k_n - 1)M_n.$$

It follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0. \tag{2.6}$$

From (1.7), we see that

$$\|y_n\| \to \|\bar{x}\| \quad \text{as } n \to \infty. \tag{2.7}$$

It follows that

$$\|Jy_n\| \to \|J\bar{x}\| \quad \text{as } n \to \infty. \tag{2.8}$$

This implies that $\{Jy_n\}$ is bounded. Note that E is reflexive and E^* is also reflexive. We may assume that $Jy_n \rightharpoonup x^* \in E^*$. By the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an $x \in E$ such that $Jx = x^*$. It follows that

$$\begin{aligned} \phi(x_{n+1}, y_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n\|^2. \end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the above equality yields that

$$\begin{aligned} 0 &\ge \|\bar{x}\|^2 - 2\langle \bar{x}, x^*\rangle + \|x^*\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx\rangle + \|Jx\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx\rangle + \|x\|^2 \\ &= \phi(\bar{x}, x). \end{aligned}$$

That is, $\bar{x} = x$, which in turn implies that $x^* = J\bar{x}$. It follows that $Jy_n \rightharpoonup J\bar{x} \in E^*$. Since (2.8) and E^* satisfies the Kadec–Klee property, we obtain that

$$Jy_n - J\bar{x} \to 0 \quad \text{as } n \to \infty.$$

Note that $J^{-1} : E^* \to E$ is demi-continuous. It follows that $y_n \rightharpoonup \bar{x}$. Since (2.7) and E satisfy the Kadec–Klee property, we obtain that

$$y_n \to \bar{x} \quad \text{as } n \to \infty.$$

Notice that

$$\|x_n - y_n\| \le \|x_n - \bar{x}\| + \|\bar{x} - y_n\|.$$

It follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{2.9}$$

Notice that

$$\|Jy_n - Jx_n\| = (1 - \alpha_n)\|JT^n x_n - Jx_n\|.$$

From the assumption on $\{\alpha_n\}$ and (2.9), we see that

$$\lim_{n\to\infty} \|Jx_n - JT^n x_n\| = 0. \tag{2.10}$$

On the other hand, we have

$$\lim_{n\to\infty} \|Jx_n - J\bar{x}\| = 0. \tag{2.11}$$

In view of

$$\|JT^n x_n - J\bar{x}\| \le \|JT^n x_n - Jx_n\| + \|Jx_n - J\bar{x}\|,$$

we see from (2.10) and (2.11) that

$$\lim_{n\to\infty} \|JT^n x_n - J\bar{x}\| = 0. \tag{2.12}$$

The demi-continuity of $J^{-1} : E^* \to E$ implies that $T^n x_n \rightharpoonup \bar{x}$. Notice that

$$|\,\|T^n x_n\| - \|\bar{x}\| = |\,\|JT^n x_n\| - \|J\bar{x}\| \le \|JT^n x_n - J\bar{x}\|.$$

From (2.12), we see that $\|T^n x_n\| \to \|\bar{x}\|$ as $n \to \infty$. Since E has the Kadec–Klee property, we obtain that

$$\lim_{n\to\infty} \|T^n x_n - \bar{x}\| = 0. \tag{2.13}$$

Since

$$\|T^{n+1} x_n - \bar{x}\| \le \|T^{n+1} x_n - T^n x_n\| + \|T^n x_n - \bar{x}\|.$$

It follows from the asymptotic regularity of T and (2.13) that

$$\lim_{n\to\infty} \|T^{n+1} x_n - \bar{x}\| = 0.$$

That is, $TT^n x_n - \bar{x} \to 0$ as $n \to \infty$. It follows from the closedness of T that $T\bar{x} = \bar{x}$.

Finally, we show that $\bar{x} = \Pi_{F(T)} x_0$. From $x_n = \Pi_{C_n} x_0$, we have

$$\langle x_n - w, Jx_0 - Jx_n\rangle \ge 0, \quad \forall w \in F(T) \subset C_n. \tag{2.14}$$

Taking the limit as $n \to \infty$ in (2.14), we obtain that

$$\langle \bar{x} - w, Jx_0 - J\bar{x}\rangle \ge 0, \quad \forall w \in F(T),$$

and hence $\bar{x} = \Pi_{F(T)} x_0$ by Lemma 1.4. This completes the proof. □

Remark 2.2. Theorem 2.1 is a version of Theorem KX in Banach spaces. The hybrid projection algorithm considered in Theorem 2.1 is simpler than Theorem KX, because we can remove the set "Q_n".

Remark 2.3. If we suppose that T is Lipschitz continuous, then the assumption that T is closed and asymptotically regular can be removed.

From the definition of quasi-ϕ-nonexpansive mappings, we see that every quasi-ϕ-nonexpansive mapping is asymptotically quasi-ϕ-nonexpansive with the constant sequence $\{1\}$. From the proof of Theorem 2.1, we have the following results immediately.

Corollary 2.4. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec–Klee property and C a nonempty closed convex subset of E. Let $T : C \to C$ be a closed quasi-ϕ-nonexpansive mapping. Let $\{x_n\}$ be a sequence generated in the following manner.*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n) JTx_n], \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0. \end{cases}$$

Assume that $\{\alpha_n\}$ is a sequence in $[0, 1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from E onto $F(T)$.

Remark 2.5. Corollary 2.4 is a version of Theorem TTK in Banach spaces.

Remark 2.6. Corollary 2.4 improves Theorem MT in the following senses.

(a) We can compute the algorithm in Corollary 2.4 more easily than the one in Theorem MT without the set "W_n".
(b) Since T is a quasi-ϕ-nonexpansive mapping, we remove the restriction $F(T) = \widetilde{F}(T)$.

(c) Every uniformly convex Banach spaces must satisfy the Kadec–Klee property. So the uniformly smooth and strictly convex Banach spaces with the Kadec–Klee property are more general than the uniformly smooth and uniformly convex Banach spaces.

In a real Hilbert space H, Theorem 2.1 is reduced to the following result.

Corollary 2.7. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a closed and asymptotically quasi-nonexpansive mapping. Assume that T is asymptotically regular on C and $F(T)$ is bounded. Let $\{x_n\}$ be a sequence generated in the following manner.*

$$\begin{cases} x_0 \in H \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\| + (k_n - 1)M_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 0, \end{cases}$$

where $M_n = \sup\{\|z - x_n\|^2 : \forall z \in F(T)\}$. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Remark 2.8. If the mapping T in Corollary 2.7 is asymptotically nonexpansive, then the restriction that T is closed and asymptotically regular on C can be removed. Therefore, we have the following.

Corollary 2.9. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a asymptotically nonexpansive mapping with fixed points. Assume that $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner.*

$$\begin{cases} x_0 \in H \quad \text{chosen arbitrarily,} \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\| + (k_n - 1)M_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 0, \end{cases}$$

where $M_n = \sup\{\|z - x_n\|^2 : \forall z \in F(T)\}$. Assume that $\{\alpha_n\}$ is a sequence in $[0, 1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Remark 2.10. Corollary 2.9 improves Theorem KX in the following senses.

(d) From the point of view on computation, we remove the iterative step "Q_n" in Theorem KX.
(e) We do not assume that C is bounded as in Theorem KX, but the fixed point set of T is bounded instead.

Remark 2.11. Corollary 2.9 is reduced to Theorem TTK if the mapping T is nonexpansive.

Remark 2.12. It is of interest to improve the main results presented in this paper from a mapping to a semigroup of mappings.

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